

Weak order in averaging principle for stochastic wave equations with a fast oscillation

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Abstract

This article deals with the weak errors for averaging principle for a stochastic wave equation in a bounded interval $[0, L]$, perturbed by a oscillating term arising as the solution of a stochastic reaction-diffusion equation evolving with respect to the fast time. Under suitable conditions, it is proved that the rate of weak convergence to the averaged effective dynamics is of order 1 via an asymptotic expansion approach.

Keywords: Stochastic wave equations, averaging principle, invariant measure weak convergence, asymptotic expansion.

MSC: primary 60H15, secondary 70K70

1. Introduction

Let $D = [0, L] \subset \mathbb{R}$ be a bounded open interval. In the article, for fixed $T > 0$, we consider the following class of stochastic wave equation with fast oscillating perturbation,

$$\begin{cases} \frac{\partial^2}{\partial t^2} U_t^\epsilon(\xi) = \Delta U_t^\epsilon(\xi) + F(U_t^\epsilon(\xi), Y_{\frac{t}{\epsilon}}(\xi)) + \sigma_1 \dot{W}_t^1(\xi), & t \in [0, T], \xi \in D, \\ U_t^\epsilon(\xi) = 0, & (\xi, t) \in \partial D \times (0, T], \\ U_0^\epsilon(\xi) = x_1(\xi), \frac{\partial U_t^\epsilon(\xi)}{\partial t} \Big|_{t=0} = x_2(\xi), & \xi \in D, \end{cases} \quad (1.1)$$

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where ϵ is positive parameter, Y_t is governed by the stochastic reaction-diffusion equation:

$$\begin{cases} \frac{\partial}{\partial t} Y_t(\xi) = \Delta Y_t(\xi) + g(Y_t(\xi)) + \sigma_2 \dot{W}_t^2(\xi), & t \in [0, T], \xi \in D, \\ Y_t(\xi) = 0, & (\xi, t) \in \partial D \times (0, T], \\ Y_0(\xi) = y(\xi), \end{cases} \quad (1.2)$$

Assumptions on the smoothness of the drift f and g will be given below. The stochastic perturbations are of additive type and $W_t^1(\xi)$ and $W_t^2(\xi)$ are mutually independent $L^2(D)$ -valued Wiener processes on a complete stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, which will be specified later. The noise strength coefficients σ_1 and σ_2 are positive constants and the parameter ϵ is small, which describes the ratio of time scale between the process $X_t^\epsilon(\xi)$ and $Y_{t/\epsilon}(\xi)$. With this time scale the variable $X_t^\epsilon(\xi)$ is referred as slow component and $Y_{t/\epsilon}(\xi)$ as the fast component.

The equation (1.1) is an abstract model for a random vibration of a elastic string with a fast oscillating perturbation. More generally, the nonlinear coupled wave-heat equations with fast and slow time scales may describe a thermoelastic wave propagation in a random medium [9], the interactions of fluid motion with other forms of waves [23, 38], wave phenomena which are heat generating or temperature related [22], magneto-elasticity [26] and biological problems [8, 4, 32].

Averaging principle plays an important role in the study of asymptotic behavior for slow-fast dynamical systems. It was first studied by Bogoliubov[2] for deterministic differential equations. The theory of averaging for stochastic ordinary equations may be found in [16], the works of Freidlin and Wentzell [11, 12], Veretennikov [27, 28], and Kifer [19, 20, 21]. Further progress on averaging for stochastic dynamical systems with non-Gaussian noise in finite dimensional space was studied in [33, 34, 35, 36, 37]. Concerning the infinite dimensional case, it is worth quoting the paper by Cerrai [5, 6, 7], Bréhier [3], Wang [30], Fu [13] and Bao [1].

In our previous article [14], the asymptotic limit dynamics (as ϵ tends to 0) of system (1.1) was explored within averaging framework. Under suitable conditions, it can be shown that a reduced stochastic wave equation, without the fast component, can be constructed to characterize the essential dynamics of (1.1) in a pathwise sense, as it is done in [5, 6, 7] for stochastic partial equations of parabolic type and for stochastic ordinary differential equations [15, 24, 29].

In the present paper, we are interested in the rate of weak convergence of the averaging dynamics to the true solution of slow motion $U_t^\epsilon(\xi)$. Namely, we will determine the order, with respect to timescale parameter ϵ , of weak deviations between original solution of slow equation and the solution of the corresponding averaged equation. To our knowledge, up to now this problem has been treated only in the case of deterministic reaction diffusion equations in dimension $d = 1$ subjected with a random perturbation evolving with respect to the fast time t/ϵ (to this purpose we refer to the paper by Bréhier [3]). Once the noise is included in slow variable, the method in [3] used to obtain the weak order $1 - \epsilon$

for arbitrarily small $\varepsilon > 0$ will be more complicated due to the lack of time regularity for slow solution.

In the situation we are considering, an additive time-space white noise is included in the slow motion and the main results show that order 1 for weak convergence can be derived, which can be compared with the order $1 - \varepsilon$ in [3]. Under dissipative assumption on Eq. (1.2), the perturbation process Y_t admits a unique invariant measure μ with mixing property. Then, by averaging the drift coefficient of the slow motion Eq. (1.1) with respect to the invariant measure μ , the effective equation with following form can be established:

$$\begin{cases} \frac{\partial^2}{\partial t^2} \bar{U}_t(\xi) = \Delta \bar{U}_t(\xi) + \bar{F}(\bar{U}_t(\xi)) + \sigma_1 \dot{W}_t^1(\xi), \\ \bar{U}_t(\xi) = 0, (\xi, t) \in \partial D \times (0, T], \\ \bar{U}_0(\xi) = x_1(\xi), \frac{\partial \bar{U}_t(\xi)}{\partial t} \Big|_{t=0} = x_2(\xi), \xi \in D, \end{cases}$$

where for any $u, y \in H := L^2(D)$,

$$\bar{F}(u) := \int_H F(u, y) \mu(dy), u \in H.$$

We prove that, under a smoothness assumption on drift coefficient in the slow motion equation, an error estimate of the following form

$$|\mathbb{E}\phi(U_t^\varepsilon) - \mathbb{E}\phi(\bar{U}_t)| \leq C\varepsilon$$

for any function ϕ with derivatives bounded up to order 3. In order to prove the validity of above bound, we adopt asymptotic expansion schemes in [3] to decompose $\mathbb{E}\phi(U_t^\varepsilon)$ with respect to the scale parameter ε in form of

$$\mathbb{E}\phi(U_t^\varepsilon) = u_0 + \varepsilon u_1 + r^\varepsilon,$$

where the functions u_0 has to coincide with $\mathbb{E}\phi(\bar{U}_t)$ by uniqueness discuss, as it can be shown that they are governed by the same Kolmogorov equation via identification the powers of ε . Due to solvability of the Poisson equation associated with generator of perturbation process Y_t , an explicit expression of u_1 can be constructed such that its boundedness is based on a priori estimates for the Y_t and smooth dependence on initial data for averaging equation. The next step consist in identifying r^ε as the solution of a evolutionary equation and showing that $|r^\varepsilon| \leq C\varepsilon$. The proof of bound for r^ε is based on estimates on $\frac{du}{dt}$ and $\mathcal{L}_2 u_1$, where \mathcal{L}_2 is the Kolmogorov operator associated with the slow motion equation. We would like to stress that this procedure is quite involved, as it concerns a system with noise in infinite dimensional space, and the diffusion term leading to quantitative analysis on higher order differentiability of $\mathbb{E}\phi(\bar{U}_t)$ with respect to the initial datum. Let us also remark that asymptotic expansion of the solutions of Kolmogorov equations was studied in [17, 18] and [31].

The rest of the paper is arranged as follows. Section 2 is devoted to the general notation and framework. The ergodicity of fast process and the averaging dynamics of system (1.1) is introduced in Section 3. Then the main results of

this article, which is derived via the asymptotic expansions and uniform error estimates, is presented in Section 4. In the final section, we state and prove technical lemmas applied in the preceding section.

Throughout the paper, the letter C below with or without subscripts will denote generic positive constants independent of ϵ , whose value may change from one line to another.

2. Preliminary

To rewrite the systems (1.1) and (1.2) as the abstract evolution equations, we present some notations and some well-known facts for later use.

For a fixed domain $D = [0, L]$, we use the abbreviation $H := L^2(D)$ for the space of square integrable real-valued functions on D . The scalar product and norm on H are denoted by $(\cdot, \cdot)_H$ and $\|\cdot\|$, respectively.

We recall the definition of the Wiener process in infinite space. For more details, see [25]. Let $\{q_{i,k}(\xi)\}_{k \in \mathbb{N}}$ be H -valued eigenvectors of a nonnegative, symmetric operator Q_i with corresponding eigenvalues $\{\lambda_{i,k}\}_{k \in \mathbb{N}}$, for $i = 1, 2$, such that

$$Q_i q_{i,k}(\xi) = \lambda_{i,k} q_{i,k}(\xi), \quad \lambda_{i,k} > 0, k \in \mathbb{N}.$$

For $i = 1, 2$, let $W_t^i(\xi)$ be an H -valued Q_i -Wiener process with operator Q_i satisfying

$$\text{Tr} Q_i = \sum_{k=1}^{+\infty} \lambda_{i,k} < +\infty.$$

Then

$$W_t^i(\xi) = \sum_{k=1}^{+\infty} \lambda_{i,k}^{\frac{1}{2}} \beta_{i,k}(t) q_{i,k}(\xi), \quad t \geq 0,$$

where $\{\beta_{i,k}(t)\}_{k \in \mathbb{N}}^{i=1,2}$ are mutually independent real-valued Brownian motions on a probability base $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. For the abbreviation, we will sometimes omit the spatial variable ξ in the sequel.

Let $\{e_k(\xi)\}_{k \in \mathbb{N}}$ denote the complete orthonormal system of eigenfunctions in H such that, for $k = 1, 2, \dots$,

$$-\Delta e_k = \alpha_k e_k, \quad e_k(0) = e_k(L) = 0,$$

with $0 < \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_k \leq \dots$. Here we would like to recall the fact that $e_k(\xi) = \sin \frac{k\pi\xi}{L}$ and $\alpha_k = -\frac{k^2\pi^2}{L^2}$ for $k = 1, 2, \dots$.

Let A be the realization in H of the Laplace operator Δ with zero Dirichlet boundary condition, which generates a strong continuous semigroups $\{E_t\}_{t \geq 0}$, defined by, for any $h \in H$,

$$E_t h = \sum_{k=1}^{+\infty} e^{-\alpha_k t} e_k \left(e_k, h \right)_H.$$

It is straightforward to check that $\{E_t\}_{t \geq 0}$ are contractive semigroups on H .

For $s \in \mathbb{R}$, we introduce the space $H^s := D((-A)^{s/2})$, which equipped with inner product

$$\langle g, h \rangle_s := \left((-A)^{\frac{s}{2}} g, (-A)^{\frac{s}{2}} h \right)_H = \sum_{k=1}^{+\infty} \alpha_k^s \left(g, e_k \right)_H \left(h, e_k \right)_H, \quad g, h \in H^s$$

and the norm

$$\|\varphi\|_s = \left\{ \sum_{k=1}^{+\infty} \alpha_k^s \left(\varphi, e_k \right)_H^2 \right\}^{\frac{1}{2}}$$

for $\varphi \in H^s$. It is obvious that $H^0 = H$ and $H^\alpha \subset H^\beta$ for $\beta \leq \alpha$. We note that in the case of $s > 0$, H^{-s} can be identified with the dual space $(H^s)^*$, i.e. the space of the linear functional on H^s which are continuous with respect to the topology induced by the norm $\|\cdot\|_s$. We shall denote by \mathcal{H}^α the product space $H^\alpha \times H^{\alpha-1}$, $\alpha \in \mathbb{R}$, endowed with the scalar product

$$\left(x, y \right)_{\mathcal{H}^\alpha} = \langle x_1, y_1 \rangle_\alpha + \langle x_2, y_2 \rangle_{\alpha-1}, \quad x = (x_1, x_2)^T, y = (y_1, y_2)^T,$$

and the corresponding norm

$$\|x\|_\alpha = \left\{ \|x_1\|_\alpha^2 + \|x_2\|_{\alpha-1}^2 \right\}^{\frac{1}{2}}, \quad x = (x_1, x_2)^T.$$

If $\alpha = 0$ we abbreviate $H^0 \times H^{-1} = \mathcal{H}$ and $\|\cdot\| = \|\cdot\|_0$. To consider (1.1)

as an abstract evolution equation, we set $V_t^\epsilon = \frac{d}{dt} U_t^\epsilon$ and let $X_t^\epsilon = \begin{bmatrix} U_t^\epsilon \\ V_t^\epsilon \end{bmatrix}$ with

$X_0^\epsilon := x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The systems (1.1) and (1.2) can be rewritten as an abstract form

$$\begin{cases} dX_t^\epsilon = \mathcal{A}X_t^\epsilon dt + \mathbf{F}(X_t^\epsilon, Y_t^\epsilon)dt + BdW_t^1, \\ dY_t^\epsilon = \frac{1}{\epsilon}AY_t^\epsilon dt + \frac{1}{\epsilon}g(Y_t^\epsilon)dt + \frac{\sigma}{\sqrt{\epsilon}}dW_t^2, \\ X_0^\epsilon = x, Y_0^\epsilon = y, \end{cases} \quad (2.1)$$

where

$$\mathcal{A} := \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}, \mathbf{F}(x, y) := \begin{bmatrix} 0 \\ F(\Pi_1 \circ x, y) \end{bmatrix}, B := \begin{bmatrix} 0 \\ I \end{bmatrix},$$

with

$$\mathcal{D}(\mathcal{A}) = \left\{ X = (x_1, x_2)^T \in \mathcal{H} : \mathcal{A}X = \begin{bmatrix} x_2 \\ Ax_1 \end{bmatrix} \in \mathcal{H} \right\} = \mathcal{H}^1,$$

here A is regarded as an operator from H^1 to H^{-1} , and Π_1 denotes the canonical projection $\mathcal{H} \rightarrow H$. It is well known that the operator \mathcal{A} is the generator of a strongly continuous semigroup $\{\mathcal{S}_t\}_{t \geq 0}$ on \mathcal{H} with the explicit form

$$\mathcal{S}_t = e^{At} = \begin{bmatrix} C(t) & (-A)^{-\frac{1}{2}}S(t) \\ -(-A)^{\frac{1}{2}}S(t) & C(t) \end{bmatrix}, \quad t \geq 0, \quad (2.2)$$

where $C(t) = \cos((-A)^{\frac{1}{2}}t)$ and $S(t) = \sin((-A)^{\frac{1}{2}}t)$ are so-called cosine and sine operators with the expression in term of the orthonormal eigenpairs $\{\alpha_i, e_i\}_{i \in \mathbb{N}}$ of A :

$$\begin{aligned} C(t)h &= \cos((-A)^{\frac{1}{2}}t)h = \sum_{k=1}^{+\infty} \cos\{\sqrt{\alpha_k}t\} \left(e_k, h \right)_H \cdot e_k, \\ S(t)h &= \sin((-A)^{\frac{1}{2}}t)h = \sum_{k=1}^{+\infty} \sin\{\sqrt{\alpha_k}t\} \left(e_k, h \right)_H \cdot e_k. \end{aligned}$$

Moreover, it is easy to check that $\|\mathcal{S}_t x\| \leq \|x\|$ for $t \geq 0, x \in \mathcal{H}$. In order to ensure existence and uniqueness of the perturbation process Y_t we shall assume throughout this paper that:

(Hypothesis 1) For the mapping $g : H \rightarrow H$, we require that there exists a constant $L_g > 0$ such that

$$\|g(u_1) - g(u_2)\| \leq L_g(\|u_1 - u_2\|), \quad u, v \in H. \quad (2.3)$$

moreover, we assume that $L_g < \alpha_1$.

Concerning the coefficient F we impose the following conditions:

(Hypothesis 2) For the mapping $F : H \times H \rightarrow H$, we assume that there exists a constant $L_F > 0$ such that

$$\|F(u_1, v_1) - F(u_2, v_2)\| \leq L_F(\|u_1 - u_2\| + \|v_1 - v_2\|), \quad u_1, u_2, v_1, v_2 \in H. \quad (2.4)$$

Also suppose that for any $u \in H$, the mapping $F(u, \cdot) : H \rightarrow H$ is of class C^2 , with bounded derivatives. Moreover, we require that there exists a constant L such that for any $u, v, w, y, y' \in H$ its directional derivatives are well-defined and satisfy

$$\|D_u F(u, y) \cdot w\| \leq L\|w\|, \quad (2.5)$$

$$\|D_{uu}^2 F(u, y) \cdot (v, w)\| \leq L\|v\| \cdot \|w\|. \quad (2.6)$$

$$\|[D_u F(u, y) - D_u F(u, y')] \cdot w\| \leq L\|y - y'\| \cdot \|w\| \quad (2.7)$$

$$\|D_{uu}^2 [F(u, y) - F(u, y')] \cdot (v, w)\| \leq L\|y - y'\| \cdot \|v\| \cdot \|w\|. \quad (2.8)$$

Remark 2.1. A simple example of the dirft coefficient F is given by

$$F(u, y) = F_1(u) + F_2(y),$$

here $F_1, F_2 : H \rightarrow H$ are of class C^2 with uniformly bounded derivatives up to order 2.

According to conditions (2.3) and (2.4), system (2.1) admits a unique mild solution. Namely, as discussed in [25], for any $y \in H$ there exists a unique adapted process $Y(y) \in L^2(\Omega, C([0, T]; H))$ such that

$$Y_t(y) = E_t y + \int_0^t E_{t-s} g(Y_s(y)) ds + \sigma_2 \int_0^t E_{t-s} dW_s^2, \quad (2.9)$$

By arguing as in the proof of [25], Theorem 7.2, it is possible to show that there exists a constant $C > 0$ such that

$$\mathbb{E} \|Y_t(y)\|^2 \leq C(1 + \|y\|^2), \quad t > 0, \quad (2.10)$$

and in correspondence of such $Y_t(y)$, for any $\epsilon > 0$ and $x = (x_1, x_2)^T \in \mathcal{H}$ there exists a unique adapted process $X^\epsilon(x, y) \in L^2(\Omega, C([0, T]; \mathcal{H}))$ such that

$$X_t^\epsilon(x, y) = \mathcal{S}_t x + \int_0^t \mathcal{S}_{t-s} \mathbf{F}(X_s^\epsilon(x, y), Y_{s/\epsilon}(y)) ds + \sigma_1 \int_0^t \mathcal{S}_{t-s} B dW_s^1. \quad (2.11)$$

We point out that if $x = (x_1, x_2)^T$ is taken in $D(\mathcal{A}) = \mathcal{H}^1$, then X_t^ϵ values in \mathcal{H}^1 for $t > 0$ (see [10]) and satisfies

$$\mathbb{E} \|X_t^\epsilon(x, y)\|_1^2 \leq C(1 + \|y\|^2 + \|x\|_1^2) \quad (2.12)$$

for some constant $C > 0$. Moreover, we present an estimate for the \mathcal{H} -norm of $\mathcal{A}X_t^\epsilon$, which is uniform with respect to $\epsilon > 0$.

Proposition 2.1. *Let $X_t^\epsilon(x, y) = (U_t^\epsilon(x, y), V_t^\epsilon(x, y))^T$ be the solution to the problem (2.11), where the initial value satisfies $X_0^\epsilon = x = (x_1, x_2)^T \in \mathcal{H}^1$, and the function F satisfies (2.4). Then it holds that*

$$\mathbb{E} \|\mathcal{A}X_t^\epsilon(x, y)\|^2 \leq C(1 + \|y\|^2 + \|x\|_1^2). \quad (2.13)$$

Proof. We have

$$\mathcal{A}X_t^\epsilon(x, y) = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \begin{pmatrix} U_t^\epsilon(x, y) \\ V_t^\epsilon(x, y) \end{pmatrix} = \begin{pmatrix} V_t^\epsilon(x, y) \\ A(U_t^\epsilon(x, y)) \end{pmatrix},$$

so that

$$\begin{aligned} \|\mathcal{A}X_t^\epsilon(x, y)\|^2 &= \|V_t^\epsilon(x, y)\|^2 + \|A(U_t^\epsilon(x, y))\|_{-1}^2 \\ &= \|V_t^\epsilon(x, y)\|^2 + \|A^{\frac{1}{2}}(U_t^\epsilon(x, y))\|^2. \end{aligned} \quad (2.14)$$

Let us start to estimate the norm of $A^{\frac{1}{2}}(U_t^\epsilon(x, y))$ and consider the expression

$$\begin{aligned} A^{\frac{1}{2}}U_t^\epsilon(x, y) &= A^{\frac{1}{2}}C(t)x_1 - S(t)x_2 - \int_0^t S(t-s)F(U_s^\epsilon(x, y), Y_{s/\epsilon}(y))ds \\ &\quad + \sigma_1 \int_0^t S(t-s)dW_s^1. \end{aligned}$$

Directly, we have

$$\|A^{\frac{1}{2}}C(t)x_1\|^2 + \|S(t)x_2\|^2 \leq C(\|x_1\|_1^2 + \|x_2\|^2). \quad (2.15)$$

In view of the assumptions on F given in (2.4), we obtain

$$\begin{aligned}
& \mathbb{E} \left\| \int_0^t S(t-s) F(U_s^\epsilon(x, y), Y_{s/\epsilon}(y)) ds \right\|^2 \\
& \leq C_1 + C_2 \int_0^t \mathbb{E} [\|U_s^\epsilon(x, y)\|^2 + \|Y_{s/\epsilon}(y)\|^2] ds \\
& \leq C_1 + C_2 \int_0^t \mathbb{E} [\|A^{\frac{1}{2}} U_s^\epsilon(x, y)\|^2 + \|Y_{s/\epsilon}(y)\|^2] ds,
\end{aligned}$$

and then, thanks to (2.10), we have

$$\begin{aligned}
& \mathbb{E} \left\| \int_0^t S(t-s) F(U_s^\epsilon(x, y), Y_{s/\epsilon}(y)) ds \right\|^2 \\
& \leq C_1(1 + \|y\|^2) + C_2 \int_0^t \mathbb{E} \|A^{\frac{1}{2}} U_s^\epsilon(x, y)\|^2 ds. \tag{2.16}
\end{aligned}$$

Notice that in view of Ito's isometry, we have

$$\mathbb{E} \left\| \int_0^t S(t-s) dW_s^1 \right\|^2 \leq C_3$$

and then, combining this estimate with (2.15) and (2.16), we have

$$\begin{aligned}
\mathbb{E} \|A^{\frac{1}{2}} U_t^\epsilon(x, y)\|^2 & \leq C_1(1 + \|y\|^2 + \|x_1\|_1^2 + \|x_2\|^2) \\
& + C_2 \int_0^t \mathbb{E} \|A^{\frac{1}{2}} U_s^\epsilon(x, y)\|^2 ds.
\end{aligned}$$

From the Gronwall's lemma, this gives

$$\mathbb{E} \|A^{\frac{1}{2}} U_t^\epsilon(x, y)\|^2 \leq C_1(1 + \|y\|^2 + \|x_1\|_1^2 + \|x_2\|^2).$$

In an analogous way, we can prove that

$$\mathbb{E} \|V_t^\epsilon(x, y)\|^2 \leq C_1(1 + \|y\|^2 + \|x_1\|_1^2 + \|x_2\|^2).$$

Thanks to (2.14), the two inequalities above yield (2.13). \square

If \mathcal{X} is a Hilbert space equipped with inner product $(\cdot, \cdot)_{\mathcal{X}}$, we denote by $C^1(\mathcal{X}, \mathbb{R})$ the space of all real function $\phi : \mathcal{X} \rightarrow \mathbb{R}$ with continuous Fréchet derivative and use the notation $D\phi(x)$ for the differential of a C^1 function on \mathcal{X} at the point x . Thanks to Riesz representation theorem, we may get the identity for $x, h \in \mathcal{X}$:

$$D\phi(x) \cdot h = (D\phi(x), h)_{\mathcal{X}}.$$

We define $C_b^2(\mathcal{X}, \mathbb{R})$ to be the space of all real-valued, twice Fréchet differential function on \mathcal{X} , whose first and second derivatives are continuous and bounded. For $\phi \in C_b^2(\mathcal{X}, \mathbb{R})$, we will identify $D^2\phi(x)$ with a bilinear operator from $\mathcal{X} \times \mathcal{X}$ to \mathbb{R} such that

$$D^2\phi(x) \cdot (h, k) = (D^2\phi(x)h, k)_{\mathcal{X}}, \quad x, h, k \in \mathcal{X}.$$

On some occasions, we also use the notation ϕ', ϕ'' instead of $D\phi$ or $D^2\phi$.

3. Ergodicity of Y_t and averaging dynamics

Now, we consider the transition semigroup P_t associated with perturbation process $Y_t(y)$ defined by equation (2.9), by setting for any $\psi \in \mathcal{B}_b(H)$ the space of bounded functions on H ,

$$P_t\psi(y) = \mathbb{E}\psi(Y_t(y)).$$

By arguing as [13], we can show that

$$\mathbb{E}\|Y_t(y)\|^2 \leq C \left(e^{-(\alpha_1 - L_g)t} \|y\|^2 + 1 \right), \quad t > 0 \quad (3.1)$$

for some constant $C > 0$. This implies that there exists an invariant measure μ for the Markov semigroup P_t associated with system (2.9) in H such that

$$\int_H P_t\psi d\mu = \int_H \psi d\mu, \quad t \geq 0$$

for any $\psi \in \mathcal{B}_b(H)$ (for a proof, see, e.g., [6], Section 2.1). Then by repeating the standard argument as in the proof of Proposition 4.2 in [7], the invariant measure has finite 2-moments:

$$\int_H \|y\|^2 \mu(dy) \leq C. \quad (3.2)$$

Let $Y_t(y')$ be the solution of (2.9) with initial value $Y_0 = y'$, it can be check that for any $t \geq 0$,

$$\mathbb{E}\|Y_t(y) - Y_t(y')\|^2 \leq \|y - y'\|^2 e^{-\eta t} \quad (3.3)$$

with $\eta = (\alpha_1 - L_g) > 0$, which implies that μ is the unique invariant measure for P_t . Then, by averaging the coefficient F with respect to the invariant measure μ , we can define a H -valued mapping

$$\bar{F}(u) := \int_H F(u, y) \mu(dy), \quad u \in H,$$

and then, due to condition (2.4), it is easily to check that

$$\|\bar{F}(u_1) - \bar{F}(u_2)\| \leq L \|u_1 - u_2\|, \quad u_1, u_2 \in H. \quad (3.4)$$

Now we will consider the effective dynamics system

$$\begin{cases} \frac{\partial^2}{\partial t^2} \bar{U}_t(\xi) = \Delta \bar{U}_t(\xi) + \bar{F}(\bar{U}_t(\xi)) + \sigma_1 \dot{W}_t^1, & (\xi, t) \in D \times [0, T], \\ \bar{U}_t(\xi) = 0, & (\xi, t) \in \partial D \times [0, +\infty), \\ \bar{U}_0(\xi) = x_1(\xi), \quad \frac{\partial}{\partial t} \bar{U}_t(\xi)|_{t=0} = x_2(\xi), & \xi \in D. \end{cases} \quad (3.5)$$

Following the same notation as in Section 2, the problem (3.5) can be transferred to a stochastic evolution equation:

$$\begin{cases} d\bar{X}_t = \mathcal{A}\bar{X}_t dt + \bar{\mathbf{F}}(\bar{X}_t) dt + B dW_t^1, \\ \bar{X}_0 = x, \end{cases} \quad (3.6)$$

where $\bar{X}_t = \begin{bmatrix} \bar{U}_t \\ \bar{V}_t \end{bmatrix}$ with $\bar{V}_t = \frac{d}{dt}\bar{U}_t$ and $\bar{\mathbf{F}}(x) := \begin{bmatrix} 0 \\ \bar{F}(\Pi_1 \circ x) \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{F}(u) \end{bmatrix}$.
The mild form for system (3.6) is given by

$$\bar{X}_t(x) = \mathcal{S}_t x + \int_0^t \mathcal{S}_{t-s} \bar{\mathbf{F}}(\bar{X}_s(x)) ds + \sigma_1 \int_0^t \mathcal{S}_{t-s} B dW_s^1.$$

By arguing as before, for any $x = (x_1, x_2)^T \in \mathcal{H}$ the above integral equation admits a unique mild solution in $L^2(\Omega, C([0, T]; \mathcal{H}))$ such that

$$\mathbb{E} \|\bar{X}_t(x)\| \leq C(1 + \|x\|), \quad t \in [0, T]. \quad (3.7)$$

4. Asymptotic expansions

Let $\phi \in C_b^2(H, \mathbb{R})$ and define a function $u^\epsilon : [0, T] \times \mathcal{H} \times H \rightarrow \mathbb{R}$ by

$$u^\epsilon(t, x, y) = \mathbb{E} \phi(U_t^\epsilon(x, y)).$$

Let Π_1 be the canonical projection $\mathcal{H} \rightarrow H$. Define the function $\Phi : \mathcal{H} \rightarrow \mathbb{R}$ by $\Phi(x) := \phi(\Pi_1 x) = \phi(x_1)$ for $x = (x_1, x_2)^T \in \mathcal{H}$. Clearly, we have

$$u^\epsilon(t, x, y) = \mathbb{E} \Phi(X_t^\epsilon(x, y)).$$

We now introduce two differential operators associated with the systems (2.9) and (2.11), respectively:

$$\begin{aligned} \mathcal{L}_1 \varphi(y) &= \left(Ay + g(y), D_y \varphi(y) \right)_H \\ &\quad + \frac{1}{2} \sigma_2^2 \text{Tr}(D_{yy}^2 \varphi(y) Q_2 (Q_2)^*), \quad \varphi(y) \in C_b^2(H, \mathbb{R}), \end{aligned}$$

$$\begin{aligned} \mathcal{L}_2 \Psi(x) &= \left(\mathcal{A}x + \mathbf{F}(x, y), D_x \Psi(x) \right)_\mathcal{H} \\ &\quad + \frac{1}{2} \sigma_1^2 \text{Tr}(D_{xx}^2 \Psi(x) B Q_1 (B Q_1)^*), \quad \Psi(x) \in C_b^2(\mathcal{H}, \mathbb{R}). \end{aligned}$$

It is known that u^ϵ is a solution to the forward Kolmogorov equation:

$$\begin{cases} \frac{d}{dt} u^\epsilon(t, x, y) = \mathcal{L}^\epsilon u^\epsilon(t, x, y), \\ u^\epsilon(0, x, y) = \Phi(x), \end{cases} \quad (4.1)$$

where $\mathcal{L}^\epsilon = \frac{1}{\epsilon} \mathcal{L}_1 + \mathcal{L}_2$.

Also recall the Kolmogorov operator for the averaging system is defined as

$$\begin{aligned} \bar{\mathcal{L}} \Psi(x) &= \left(\mathcal{A}x + \bar{\mathbf{F}}(x), D_x \Psi(x) \right)_\mathcal{H} \\ &\quad + \frac{1}{2} \sigma_1^2 \text{Tr}(D_{xx}^2 \Psi(x) B Q_1 (B Q_1)^*), \quad \Psi(x) \in C_b^2(\mathcal{H}, \mathbb{R}). \end{aligned}$$

If we set

$$\bar{u}(t, x) = \mathbb{E}\phi(\bar{U}_t(x)) = \mathbb{E}\Phi(\bar{X}_t(x)),$$

we have

$$\begin{cases} \frac{d}{dt}\bar{u}(t, x) = \bar{\mathcal{L}}\bar{u}(t, x), \\ \bar{u}(0, x) = \Phi(x). \end{cases} \quad (4.2)$$

Then the weak difference at time T is equal to

$$\mathbb{E}\phi(\bar{U}_T) - \mathbb{E}\phi(U_T^\epsilon) = u^\epsilon(T, x, y) - \bar{u}(T, x).$$

Henceforth, when there is no confusion, we often omit the temporal variable t and spatial variables x and y . For example, for $u^\epsilon(t, x, y)$, we often write it as u^ϵ . Our aim is to seek matched asymptotic expansions for the $u^\epsilon(T, x, y)$ of the form

$$u^\epsilon = u_0 + \epsilon u_1 + r^\epsilon, \quad (4.3)$$

where u_0 and u_1 are smooth functions which will be constructed below, and r^ϵ is the remainder term. With the above assumptions and notation we have the following result, which is a direct consequence of Lemma 4.1, Lemma 4.2 and Lemma 4.5.

Theorem 4.1. *Assume that $x \in \mathcal{H}^1, y \in D(A)$. Then, under Hypotheses 1 and 2, for any $T > 0$ and $\phi \in C_b^3(H)$, there exist a constant $C_{T, \phi, x, y}$ such that*

$$|\mathbb{E}\phi(U_T^\epsilon(x, y)) - \mathbb{E}\phi(\bar{U}_T(x))| \leq C_{T, \phi, x, y}\epsilon.$$

4.1. The leading term

Let us first determine the leading terms. Now, substituting (4.3) into (4.1) yields

$$\begin{aligned} \frac{du_0}{dt} + \epsilon \frac{du_1}{dt} + \frac{dr^\epsilon}{dt} &= \frac{1}{\epsilon} \mathcal{L}_1 u_0 + \mathcal{L}_1 u_1 + \frac{1}{\epsilon} \mathcal{L}_1 r^\epsilon \\ &\quad + \mathcal{L}_2 u_0 + \epsilon \mathcal{L}_2 u_1 + \mathcal{L}_2 r^\epsilon. \end{aligned}$$

By comparing coefficients of powers of ϵ , we obtain

$$\mathcal{L}_1 u_0 = 0, \quad (4.4)$$

$$\frac{du_0}{dt} = \mathcal{L}_1 u_1 + \mathcal{L}_2 u_0. \quad (4.5)$$

It follows from (4.4) that u_0 is independent of y , which means

$$u_0(t, x, y) = u_0(t, x).$$

We also impose the initial condition $u_0(0, x) = \Phi(x)$. Since μ is the invariant measure of a Markov process with generator \mathcal{L}_1 , we have

$$\int_H \mathcal{L}_1 u_1(t, x, y) \mu(dy) = 0,$$

which, by invoking (4.5), implies

$$\begin{aligned} \frac{du_0}{dt}(t, x) &= \int_H \frac{du_0}{dt}(t, x) \mu(dy) \\ &= \int_H \mathcal{L}_2 u_0(t, x) \mu(dy) \\ &= \left(\mathcal{A} u_0(t, x) + \int_H \mathbf{F}(x, y) \mu(dy), D_x u_0(t, x) \right)_{\mathcal{H}} \\ &\quad + \frac{1}{2} \sigma_1^2 \text{Tr}(D_{xx}^2 u_0(t, x) B Q_1 (B Q_1)^*) \\ &= \bar{\mathcal{L}} u_0(t, x), \end{aligned}$$

so that u_0 and \bar{u} satisfies the same evolution equation. By using a uniqueness argument, such u_0 has to coincide with the solution \bar{u} and we have the following lemma:

Lemma 4.1. *Assume Hypotheses 1 and 2. Then for any $x \in D(\mathcal{A})$, $y \in D(A)$ and $T > 0$, we have $u_0(T, x, y) = \bar{u}(T, x)$.*

4.2. Construction of u_1

Let us proceed to carry out the construction of u_1 . Thanks to Lemma 4.1 and (4.2), the equation (4.5) can be rewritten

$$\bar{\mathcal{L}} \bar{u} = \mathcal{L}_1 u_1 + \mathcal{L}_2 \bar{u},$$

and hence we get an elliptic equation for u_1 with form

$$\mathcal{L}_1 u_1(t, x, y) = \left(\bar{\mathbf{F}}(x) - \mathbf{F}(x, y), D_x \bar{u}(t, x) \right)_{\mathcal{H}} := -\rho(t, x, y),$$

where ρ is of class C^2 with respect to y , with uniformly bounded derivative. Moreover, it satisfies for any $t \geq 0$ and $x \in \mathcal{H}^1$,

$$\int_H \rho(t, x, y) \mu(dy) = 0.$$

For any $y \in D(A)$ and $s > 0$ we have

$$\begin{aligned} \frac{d}{ds} P_s \rho(t, x, y) &= \left(A y + g(x, y), D_y (P_s \rho(t, x, y)) \right)_H \\ &\quad + \frac{1}{2} \sigma_2^2 \text{Tr}[D_{yy}^2 (P_s \rho(t, x, y)) Q_2 Q_2^*], \end{aligned}$$

here

$$P_s \rho(t, x, y) = \mathbb{E} \rho(t, x, Y_s(y))$$

satisfying

$$\lim_{s \rightarrow +\infty} \mathbb{E} \rho(t, x, Y_s(y)) = \int_H \rho(t, x, z) \mu(dz) = 0. \quad (4.6)$$

Indeed, by the invariant property of μ and Lemma 5.3,

$$\begin{aligned} & \left| \mathbb{E} \rho(t, x, Y_s(y)) - \int_H \rho(t, x, z) \mu(dz) \right| \\ &= \left| \int_H \mathbb{E} [\rho(t, x, Y_s(y)) - \rho(t, x, Y_s(z)) \mu(dz)] \right| \\ &\leq \int_H \left| \mathbb{E} \left(\mathbf{F}(x, Y_s(z)) - \mathbf{F}(x, Y_s(y), D_x \bar{u}(t, x)) \right)_{\mathcal{H}} \right| \mu(dz) \\ &\leq C \int_H \mathbb{E} \|Y_s(z) - Y_s(y)\| \mu(dz). \end{aligned}$$

This, in view of (3.3) and (3.2), yields

$$\begin{aligned} & \left| \mathbb{E} \rho(t, x, Y_s(y)) - \int_H \rho(t, x, z) \mu(dz) \right| \\ &\leq C e^{-\frac{\eta}{2}s}, \end{aligned}$$

which implies the equality (4.6). Therefore, we get

$$\begin{aligned} & \left(Ay + g(x, y), D_y \int_0^{+\infty} P_s \rho(t, x, y) ds \right)_H \\ &+ \frac{1}{2} \sigma_2^2 \text{Tr} [D_{yy}^2 \int_0^{+\infty} (P_s \rho(t, x, y)) Q_2 Q_2^*] ds \\ &= \int_0^{+\infty} \frac{d}{ds} P_s \rho(t, x, y) ds \\ &= \lim_{s \rightarrow +\infty} \mathbb{E} \rho(t, x, Y_s(y)) - \rho(t, x, y) \\ &= \int_H \rho(t, x, y) \mu(dy) - \rho(t, x, y) \\ &= -\rho(t, x, y), \end{aligned}$$

which means $\mathcal{L}_1(\int_0^{+\infty} P_s \rho(t, x, y) ds) = -\rho(t, x, y)$. Therefore, we can set

$$u_1(t, x, y) = \int_0^{+\infty} \mathbb{E} \rho(t, x, Y_s(y)) ds. \quad (4.7)$$

Lemma 4.2. *Assume Hypotheses 1 and 2. Then for any $x \in D(\mathcal{A})$, $y \in D(A)$ and $T > 0$, we have*

$$|u_1(t, x, y)| \leq C_T (1 + \|y\|), \quad t \in [0, T]. \quad (4.8)$$

Proof. As known from (4.7), we have

$$u_1(t, x, y) = \int_0^{+\infty} \mathbb{E} \left(\bar{\mathbf{F}}(x) - \mathbf{F}(x, Y_s(y)), D_x \bar{u}(t, x) \right)_{\mathcal{H}} ds.$$

This implies that

$$|u_1(t, x, y)| \leq \int_0^{+\infty} \| \bar{\mathbf{F}}(x) - \mathbb{E}[\mathbf{F}(x, Y_s(y))] \| \cdot \| D_x \bar{u}(t, x) \| ds.$$

Then, in view of Lemma 5.3 and (5.1), this implies :

$$\begin{aligned} |u_1(t, x, y)| &\leq C_T(1 + \|y\|) \int_0^{+\infty} e^{-\frac{\eta}{2}s} ds \\ &\leq C_T(1 + \|y\|). \end{aligned}$$

□

4.3. Determination of remainder r^ϵ

Once u_0 and u_1 have been determined, we can carry out the construction of the remainder r^ϵ . It is known that

$$(\partial_t - \mathcal{L}^\epsilon)u^\epsilon = 0,$$

which, together with (4.4) and (4.5), implies

$$\begin{aligned} (\partial_t - \mathcal{L}^\epsilon)r^\epsilon &= -(\partial_t - \mathcal{L}^\epsilon)u_0 - \epsilon(\partial_t - \mathcal{L}^\epsilon)u_1 \\ &= -(\partial_t - \frac{1}{\epsilon}\mathcal{L}_1 - \mathcal{L}_2)u_0 - \epsilon(\partial_t - \frac{1}{\epsilon}\mathcal{L}_1 - \mathcal{L}_2)u_1 \\ &= \epsilon(\mathcal{L}_2u_1 - \partial_t u_1). \end{aligned}$$

In order to estimate the remainder term r^ϵ we need the following crucial lemmas.

Lemma 4.3. *Assume Hypotheses 1 and 2. Then for any $x \in D(\mathcal{A})$, $y \in D(A)$ and $T > 0$, we have*

$$\left| \frac{du_1}{dt}(t, x, y) \right| \leq C(1 + \|x\|_1)\|y\|, \quad t \in [0, T].$$

Proof. According to (4.7), we have

$$\frac{du_1}{dt}(t, x, y) = \int_0^{+\infty} \mathbb{E} \left(\bar{\mathbf{F}}(x) - \mathbf{F}(x, Y_s(y)), \frac{d}{dt} D_x \bar{u}(t, x) \right)_{\mathcal{H}} ds.$$

For any $h = (h_1, h_2)^T \in \mathcal{H}^1$,

$$\begin{aligned} D_x \bar{u}(t, x) \cdot h &= D_x [\mathbb{E} \phi(\Pi_1 \circ \bar{X}(t, x))] \\ &= \mathbb{E} [\phi'(\Pi_1 \circ \bar{X}(t, x)) \cdot (\Pi_1 \circ \eta_t^{h, x})], \end{aligned}$$

and hence

$$\begin{aligned} \frac{d}{dt}(D_x \bar{u}(t, x) \cdot h) &= \mathbb{E} \left[\phi''(\Pi_1 \circ \bar{X}(t, x)) \cdot \left(\Pi_1 \circ \eta_t^{h, x}, \frac{d}{dt}(\Pi_1 \circ \bar{X}(t, x)) \right) \right] \\ &+ \mathbb{E} \left[\phi'(\Pi_1 \circ \bar{X}(t, x)) \cdot \frac{d}{dt}(\Pi_1 \circ \eta_t^{h, x}) \right], \end{aligned}$$

so that, due to the fact of $\phi \in C_b^2(H, \mathbb{R})$, we obtain

$$\begin{aligned} \left| \frac{d}{dt}(D_x \bar{u}(t, x) \cdot h) \right| &\leq C \left[\mathbb{E} \|\Pi \circ \eta_t^{h, x}\|^2 \right]^{\frac{1}{2}} \cdot \left[\mathbb{E} \left\| \frac{d}{dt}(\Pi_1 \circ \bar{X}(t, x)) \right\|^2 \right]^{\frac{1}{2}} \\ &+ \mathbb{E} \left\| \frac{d}{dt}(\Pi_1 \circ \eta_t^{h, x}) \right\| \\ &\leq C \|h\|_1 \cdot \left[\mathbb{E} \left\| \frac{d}{dt}(\Pi_1 \circ \bar{X}(t, x)) \right\|^2 \right]^{\frac{1}{2}} \\ &+ \mathbb{E} \left\| \frac{d}{dt}(\Pi_1 \circ \eta_t^{h, x}) \right\|, \end{aligned} \tag{4.9}$$

where we used the estimate (5.8) such that

$$\|\Pi_1 \circ \eta_t^{h, x}\| \leq C \|h\| \leq C \|h\|_1.$$

Now, as $\bar{X}_t(x)$ is the mild solution of averaging equation with initial data $x = (x_1, x_2)^T \in \mathcal{H}^1$, we have

$$\begin{aligned} \Pi_1 \circ \bar{X}_t(x) &= \bar{U}_t(x) \\ &= C(t)x_1 + (-A)^{-\frac{1}{2}}S(t)x_2 + \int_0^t (-A)^{-\frac{1}{2}}S(t-s)\bar{F}(\bar{U}_s(x))ds \\ &+ \sigma_1 \int_0^t (-A)^{-\frac{1}{2}}S(t-s)dW_s^1 \end{aligned}$$

with

$$\begin{aligned} \frac{d}{dt}[\Pi_1 \circ \bar{X}_t(x)] &= -(-A)^{\frac{1}{2}}S(t)x_1 + C(t)x_2 + \int_0^t C(t-s)\bar{F}(\bar{U}_s(x))ds \\ &+ \sigma_1 \int_0^t C(t-s)dW_s^1, \end{aligned}$$

By straightforward computation, we have

$$\| -(-A)^{\frac{1}{2}}S(t)x_1 \|^2 \leq \|x_1\|_1^2 \tag{4.10}$$

and

$$\|C(t)x_2\|^2 \leq \|x_2\|^2. \tag{4.11}$$

According to the Lipschitz continuity of \bar{F} and (3.7), we have

$$\begin{aligned}\mathbb{E}\left\|\int_0^t C(t-s)\bar{F}(\bar{U}_s(x))ds\right\|^2 &\leq C_T\mathbb{E}\int_0^t (1+\|\bar{U}_s(x)\|^2)ds \\ &\leq C_T(1+\|x\|_1).\end{aligned}\quad (4.12)$$

Now, from (4.10)-(4.12) it follows

$$\mathbb{E}\left\|\frac{d}{dt}[\Pi_1 \circ \bar{X}_t(x)]\right\|^2 \leq C(1+\|x\|_1^2) \quad (4.13)$$

Now, we prove uniform bounds for time derivative of $\Pi_1 \circ \eta_t^{h,x}$ with respect to x . Clearly, we have

$$\begin{aligned}\frac{d}{dt}(\Pi_1 \circ \eta_t^{h,x}) &= -(-A)^{\frac{1}{2}}S(t)h_1 + C(t)h_2 \\ &\quad + \int_0^t C(t-s)[\Pi_1 \circ (\bar{\mathbf{F}}'(\bar{X}_s(x)) \cdot \eta_t^{h,x})]ds.\end{aligned}$$

Note that for any $h = (h_1, h_2)^T \in \mathcal{H}^1$,

$$\|(-A)^{\frac{1}{2}}S(t)h_1\|^2 + \|C(t)h_2\|^2 \leq \|h\|_1^2. \quad (4.14)$$

In view of (5.5) and (5.8), we obtain

$$\begin{aligned}&\left\|\int_0^t C(t-s)[\Pi_1 \circ (\bar{\mathbf{F}}'(\bar{X}_s(x)) \cdot \eta_t^{h,x})]ds\right\| \\ &\leq C\int_0^t \|\eta_s^{h,x}\|ds \\ &\leq C_T\|h\|_1.\end{aligned}\quad (4.15)$$

Then thanks to (4.14) and (4.15), we obtain

$$\left\|\frac{d}{dt}(\Pi_1 \circ \eta_t^{h,x})\right\|^2 \leq C\|h\|_1^2.$$

So, if we plug the above estimate and estimate (4.13) into (4.9), we get

$$\left|\frac{d}{dt}D_x\bar{u}(t, x) \cdot h\right| \leq C\|h\|_1(1+\|x\|_1),$$

which, together with (5.1), implies

$$\begin{aligned}\left|\frac{du_1}{dt}(t, x, y)\right| &\leq C(1+\|x\|_1)\int_0^{+\infty} \|\bar{\mathbf{F}}(x) - \mathbb{E}\mathbf{F}(x, Y_s(y))\|_1 ds \\ &\leq C(1+\|x\|_1)\|y\|\int_0^{+\infty} e^{-\frac{\eta}{2}s} ds \\ &\leq C(1+\|x\|_1)\|y\|.\end{aligned}$$

Hence the assertions is completely proved. \square

Lemma 4.4. *Assume that all conditions in Lemma 4.3 are fulfilled. Then we have*

$$|\mathcal{L}_2 u_1(t, x, y)| \leq (1 + \|\mathcal{A}x\| + \|x\|_1 + \|y\|)(1 + \|y\|), \quad t \in [0, T].$$

Proof. As known, for any $x \in D(\mathcal{A})$,

$$\begin{aligned} \mathcal{L}_2 u_1(t, x, y) &= \left(\mathcal{A}x + \mathbf{F}(x, y), D_x u_1(t, x, y) \right)_{\mathcal{H}} \\ &+ \frac{1}{2} \sigma_2^2 Tr \left(D_{xx}^2 u_1(t, x, y) (BQ_1) (BQ_1)^* \right). \end{aligned}$$

We will carry out the estimate of $|\mathcal{L}_2 u_1(t, x, y)|$ in two steps.

(Step 1) Estimate of $\left(\mathcal{A}x + \mathbf{F}(x, y), D_x u_1(t, x, y) \right)_{\mathcal{H}}$.

For any $k \in \mathcal{H}$, we have

$$\begin{aligned} D_x u_1(t, x, y) \cdot k &= \int_0^{+\infty} \mathbb{E} \left(D_x (\bar{\mathbf{F}}(x) - \mathbf{F}(x, Y_s)) \cdot k, D_x \bar{u}(t, x) \right)_{\mathcal{H}} ds \\ &+ \int_0^{+\infty} \mathbb{E} \left(\bar{\mathbf{F}}(x) - \mathbf{F}(x, Y_s), D_{xx}^2 \bar{u}(t, x) \cdot k \right)_{\mathcal{H}} ds \\ &:= I_1(t, x, y, k) + I_2(t, x, y, k). \end{aligned}$$

According to the invariant property of measure μ , (5.4) and (2.7) we have

$$\begin{aligned} &|I_1(t, x, y, k)| \\ &\leq \int_0^{+\infty} \left| \mathbb{E} \left(D_x (\bar{\mathbf{F}}(x) - \mathbf{F}(x, Y_s(y))) \cdot k, D_x \bar{u}(t, x) \right)_{\mathcal{H}} \right| ds \\ &= \int_0^{+\infty} \left| \mathbb{E} \int_H \left(D_x [\mathbf{F}(x, z) - \mathbf{F}(x, Y_s(y))] \cdot k, D_x \bar{u}(t, x) \right)_{\mathcal{H}} \mu(dz) \right| ds \\ &= \int_0^{+\infty} \left| \mathbb{E} \int_H \left(D_x [\mathbf{F}(x, Y_s(z)) - \mathbf{F}(x, Y_s(y))] \cdot k, D_x \bar{u}(t, x) \right)_{\mathcal{H}} \mu(dz) \right| ds \\ &\leq C \|k\| \cdot \|D_x \bar{u}(t, x)\| \cdot \int_0^{+\infty} \left[\int_H \mathbb{E} \|Y_s(z) - Y_s(y)\| \mu(dz) \right] ds \end{aligned}$$

By making use of (3.3) and (3.2), the above yields

$$\begin{aligned} |I_1(t, x, y, k)| &\leq C \|k\| \cdot \|D_x \bar{u}(t, x)\| \cdot \int_0^{+\infty} e^{-\frac{\eta}{2}s} (1 + \|y\|) ds \\ &\leq C \|k\| \cdot \|D_x \bar{u}(t, x)\| \\ &\leq C \|k\|, \end{aligned} \tag{4.16}$$

where we used Lemma 5.3 in the last step. By Lemma 5.4 and (5.1), we have

$$|I_2(t, x, y, k)| \leq \int_0^{+\infty} \left| \left(\bar{\mathbf{F}}(x) - \mathbb{E} \mathbf{F}(x, Y_s(y)), D_{xx}^2 \bar{u}(t, x) \cdot k \right)_{\mathcal{H}} \right| ds$$

$$\begin{aligned}
&\leq C\|k\| \int_0^{+\infty} \|\bar{\mathbf{F}}(x) - \mathbb{E}\mathbf{F}(x, Y_s(y))\| ds \\
&\leq C\|k\|(1 + \|y\|) \int_0^{+\infty} e^{-\frac{\eta}{2}s} ds \\
&\leq C\|k\|(1 + \|y\|).
\end{aligned}$$

Together with (4.16), this yields

$$|D_x u_1(t, x, y) \cdot k| \leq C\|k\|(1 + \|y\|)$$

which means

$$\begin{aligned}
&\left| \left(\mathcal{A}x + \mathbf{F}(x, y), D_x u_1(t, x, y) \right)_{\mathcal{H}} \right| \\
&\leq C(1 + \|\mathcal{A}x\| + \|x\|_1 + \|y\|)(1 + \|y\|). \tag{4.17}
\end{aligned}$$

(Step 2) Estimate of $Tr\left(D_{xx}^2 u_1(t, x, y)(BQ_1)(BQ_1)^*\right)$. Note that we have

$$\begin{aligned}
&D_{xx} u_1(t, x, y) \cdot (h, k) \\
&= \int_0^{+\infty} \mathbb{E} \left(D_{xx}^2 (\bar{\mathbf{F}}(x) - \mathbf{F}(x, Y_s(y))) \cdot (h, k), D_x \bar{u}(t, x) \right)_{\mathcal{H}} ds \\
&+ \int_0^{+\infty} \mathbb{E} \left(D_x (\bar{\mathbf{F}}(x) - \mathbf{F}(x, Y_s(y))) \cdot h, D_{xx}^2 \bar{u}(t, x) \cdot k \right)_{\mathcal{H}} ds \\
&+ \int_0^{+\infty} \mathbb{E} \left(D_x (\bar{\mathbf{F}}(x) - \mathbf{F}(x, Y_s(y))) \cdot k, D_{xx}^2 \bar{u}(t, x) \cdot h \right)_{\mathcal{H}} ds \\
&+ \int_0^{+\infty} \mathbb{E} \left(\bar{\mathbf{F}}(x) - \mathbf{F}(x, Y_s(y)), D_{xxx}^3 \bar{u}(t, x) \cdot (h, k) \right)_{\mathcal{H}} ds \\
&:= \sum_{i=1}^4 J_i(t, x, y, h, k).
\end{aligned}$$

In view of (5.6) and invariant property of measure μ we have

$$\begin{aligned}
&|J_1(t, x, y, h, k)| \\
&\leq \int_0^{+\infty} \left| \mathbb{E} \left(D_{xx}^2 (\bar{\mathbf{F}}(x) - \mathbf{F}(x, Y_s(y))) \cdot (h, k), D_x \bar{u}(t, x) \right)_{\mathcal{H}} \right| ds \\
&= \int_0^{+\infty} \left| \mathbb{E} \int_H \left(D_{xx}^2 [\mathbf{F}(x, z) - \mathbf{F}(x, Y_s(y))] \cdot (h, k), D_x \bar{u}(t, x) \right)_{\mathcal{H}} \mu(dz) \right| ds \\
&= \int_0^{+\infty} \left| \mathbb{E} \int_H \left(D_{xx}^2 [\mathbf{F}(x, Y_s(z)) - \mathbf{F}(x, Y_s(y))] \cdot (h, k), D_x \bar{u}(t, x) \right)_{\mathcal{H}} \mu(dz) \right| ds
\end{aligned}$$

By taking (2.8) and Lemma 5.3 into account, we can deduce

$$|J_1(t, x, y, h, k)|$$

$$\begin{aligned}
&\leq C \|h\| \cdot \|k\| \cdot \int_0^{+\infty} \left[\int_H \mathbb{E} \|Y_s(z) - Y_s(y)\| \mu(dz) \right] ds \\
&\leq C \|h\| \cdot \|k\| \cdot \int_0^{+\infty} e^{-\frac{\eta}{2}s} (1 + \|y\|) ds \\
&\leq C \|h\| \cdot \|k\| \\
&\leq C \|h\| \cdot \|k\| (1 + \|y\|),
\end{aligned} \tag{4.18}$$

Again, by (5.4) and invariant property of measure μ , we have

$$\begin{aligned}
&|J_2(t, x, y, h, k)| \\
&\leq \int_0^{+\infty} \left| \mathbb{E} \int_H \left(D_x [\mathbf{F}(x, Y_s(z)) - \mathbf{F}(x, Y_s(y))] \cdot h, D_{xx}^2 \bar{u}(t, x) \cdot k \right)_{\mathcal{H}} \mu(dz) \right| ds,
\end{aligned}$$

which, by Lemma 5.4 and condition (2.7), implies

$$\begin{aligned}
&|J_2(t, x, y, h, k)| \\
&\leq C \|h\| \cdot \|k\| \cdot \int_0^{+\infty} \left[\int_H \mathbb{E} \|Y_s(z) - Y_s(y)\| \mu(dz) \right] ds \\
&\leq C \|h\| \cdot \|k\| \cdot \int_0^{+\infty} e^{-\frac{\eta}{2}s} (1 + \|y\|) ds \\
&\leq C \|h\| \cdot \|k\| (1 + \|y\|).
\end{aligned} \tag{4.19}$$

Parallel to (4.19), we can obtain the same estimate for $J_3(t, x, y, h, k)$, that is,

$$|J_3(t, x, y, h, k)| \leq C \|h\| \cdot \|k\| (1 + \|y\|). \tag{4.20}$$

By proceeding again as in the estimate for $J_1(t, x, y, h, k)$ we have

$$\begin{aligned}
&|J_4(t, x, y, h, k)| \\
&\leq \int_0^{+\infty} \left| \mathbb{E} \int_H \left(\mathbf{F}(x, Y_s(z)) - \mathbf{F}(x, Y_s(y)), D_{xxx}^3 \bar{u}(t, x) \cdot (h, k) \right)_{\mathcal{H}} \mu(dz) \right| ds
\end{aligned}$$

and then thanks to Lemma 5.5 and (2.4), we get

$$\begin{aligned}
&|J_4(t, x, y, h, k)| \\
&\leq C \|h\| \cdot \|k\| \cdot \int_0^{+\infty} \left[\int_H \mathbb{E} \|Y_s(z) - Y_s(y)\| \mu(dz) \right] ds \\
&\leq C \|h\| \cdot \|k\| (1 + \|y\|).
\end{aligned} \tag{4.21}$$

Collecting together (4.18), (4.19), (4.20) and (4.21), we obtain

$$|D_{xx}^2 u_1(t, x, y) \cdot (h, k)| \leq C \|h\| \cdot \|k\| (1 + \|y\|),$$

which means that for fixed $y \in H$ and $t \in [0, T]$,

$$\|D_{xx}^2 u_1(t, \cdot, y)\|_{L(\mathcal{H} \times \mathcal{H}, \mathbb{R})} \leq C(1 + \|y\|),$$

so that, as the operator Q_1 has finite trace, we get

$$\begin{aligned} & \text{Tr}\left(D_{xx}^2 u_1(t, x, y)(BQ_1)(BQ_1)^*\right) \\ & \leq \|D_{xx}^2 u_1(t, x, y)\| \text{Tr}\left((BQ_1)(BQ_1)^*\right) \\ & \leq C(1 + \|y\|). \end{aligned} \quad (4.22)$$

Finally, by taking inequalities (4.17) and (4.22) into account, we can conclude the proof of the lemma. \square

As a consequence of Lemma 4.3 and 4.4, we have the following fact for the remainder term r^ϵ .

Lemma 4.5. *Under the conditions of Lemma 4.3, for any $T > 0$, $x \in D(\mathcal{A})$, $y \in H$, we have*

$$|r^\epsilon(T, x, y)| \leq C\epsilon(1 + \|x\| + \|y\|)(1 + \|\mathcal{A}x\| + \|x\|_1).$$

Proof. By a variation of constant formula, we have

$$\begin{aligned} r^\epsilon(T, x, y) &= \mathbb{E}[r^\epsilon(0, X_T^\epsilon(x, y), Y_{T/\epsilon}(y))] \\ &+ \epsilon \mathbb{E}\left[\int_0^T (\mathcal{L}_2 u_1 - \frac{\partial u_1}{\partial s})(X_{T-s}^\epsilon(x, y), Y_{\frac{T-s}{\epsilon}}(y)) ds\right]. \end{aligned}$$

Since u^ϵ and $u_0 = \bar{u}$ has the same initial condition $\Phi(x)$, we have

$$\begin{aligned} |r^\epsilon(0, x, y)| &= |u^\epsilon(0, x, y) - \bar{u}(0, x) - \epsilon u_1(0, x, y)| \\ &= \epsilon |u_1(0, x, y)|, \end{aligned}$$

so that, from (4.8) and (2.10) we have

$$\mathbb{E}[r^\epsilon(0, X_T^\epsilon(x, y), Y_{T/\epsilon}(y))] \leq C\epsilon(1 + \|y\|). \quad (4.23)$$

Thanks to Lemma 4.3 and Lemma 4.4, we have

$$\begin{aligned} & \mathbb{E}[(\mathcal{L}_2 u_1 - \frac{\partial u_1}{\partial s})(X_{T-s}^\epsilon(x, y), Y_{\frac{T-s}{\epsilon}}(y))] \\ & \leq C\mathbb{E}[(1 + \|\mathcal{A}X_{T-s}^\epsilon(x, y)\| + \|X_{T-s}^\epsilon(x, y)\|_1 + \|Y_{\frac{T-s}{\epsilon}}(y)\|) \\ & \quad \cdot (1 + \|Y_{\frac{T-s}{\epsilon}}(y)\|)], \end{aligned}$$

and, according to (2.10), (2.12), (2.13) and the Hölder inequality, this implies that

$$\begin{aligned} & \mathbb{E}\left[\int_0^T (\mathcal{L}_2 u_1 - \frac{\partial u_1}{\partial s})(X_{T-s}^\epsilon(x, y), Y_{\frac{T-s}{\epsilon}}) ds\right] \\ & \leq C(1 + \|y\|)(1 + \|\mathcal{A}x\| + \|x\|_1 + \|y\|). \end{aligned}$$

This, together with (4.23), implies

$$|r^\epsilon(T, x, y)| \leq C\epsilon(1 + \|y\|)(1 + \|\mathcal{A}x\| + \|x\|_1 + \|y\|),$$

which completes the proof. \square

5. Appendix

In this section, we state and prove some technical lemmas needed in the former sections.

Lemma 5.1. *For any $x \in \mathcal{H}$ and $y \in H$, there exists a constant $C > 0$ such that*

$$\|\bar{\mathbf{F}}(x) - \mathbb{E}[\mathbf{F}(x, Y_t(y))]\|_1^2 \leq Ce^{-\eta t} (1 + \|y\|^2), \quad (5.1)$$

where $\eta = \alpha_1 - L_g > 0$.

Proof. According to the invariant property of μ , (3.2) and hypothesis (2.4), we have

$$\begin{aligned} \|\bar{\mathbf{F}}(x) - \mathbb{E}[\mathbf{F}(x, Y_t(y))]\|_1^2 &= \|\bar{F}(\Pi_1 x) - \mathbb{E}[F(\Pi_1 x, Y_t(y))]\|^2 \\ &= \left\| \int_H F(\Pi_1 x, z) \mu(dz) - \mathbb{E}[F(\Pi_1 x, Y_t(y))] \right\|^2 \\ &= \left\| \int_H \mathbb{E}[F(\Pi_1 x, Y_t(z)) - F(\Pi_1 x, Y_t(y))] \mu(dz) \right\|^2, \end{aligned}$$

so thanks to (3.2) and (3.3), we have

$$\begin{aligned} \|\bar{\mathbf{F}}(x) - \mathbb{E}[\mathbf{F}(x, Y_t(y))]\|_1^2 &\leq C \int_H \mathbb{E} \|Y_t(y) - Y_t(z)\|^2 \mu(dz) \\ &\leq Ce^{-\eta t} \int_H \|y - z\|^2 \mu(dz) \\ &\leq Ce^{-\eta t} (1 + \|y\|^2). \end{aligned}$$

□

Next, we introduce the following regularity results of averaging function \bar{F} .

Lemma 5.2. *For any $w \in H$, the function $(\bar{F}(\cdot), w)_H : H \rightarrow \mathbb{R}$ is Gâteaux differential and for any $v \in H$, it hold that*

$$(D\bar{F}(u) \cdot v, w)_H = \int_H (D_u F(u, y) \cdot v, w)_H \mu(dy), \quad w \in H.$$

Proof. For any $\lambda \neq 0$ we have

$$\begin{aligned} &\left(\int_H \frac{1}{\lambda} [F(u + \lambda v, y) - F(u, y)] \mu(dy), w \right)_H - \int_H (D_u F(u, y) \cdot v, w)_H \mu(dy) \\ &= \int_H \left(\frac{1}{\lambda} [F(u + \lambda v, y) - F(u, y)] - D_u F(u, y) \cdot v, w \right)_H \mu(dy). \end{aligned}$$

and then

$$\left| \left(\int_H \frac{1}{\lambda} [F(u + \lambda v, y) - F(u, y)] \mu(dy), w \right)_H - \int_H (D_u F(u, y) \cdot v, w)_H \mu(dy) \right|$$

$$\begin{aligned}
&\leq \int_H \left| \left(\frac{1}{\lambda} [F(u + \lambda v, y) - F(u, y)] - D_u F(u, y) \cdot v, w \right)_H \right| \mu(dy) \\
&\leq \|w\| \int_H \left\| \frac{1}{\lambda} [F(u + \lambda v, y) - F(u, y)] - D_u F(u, y) \cdot v \right\| \mu(dy). \tag{5.2}
\end{aligned}$$

Now, since $F(\cdot, y) : H \rightarrow H$ is Gâteaux differentiable in H , for any $h \in H$, we obtain

$$\lim_{\lambda \rightarrow 0} \left\| \frac{1}{\lambda} [F(u + \lambda v, y) - F(u, y)] - D_u F(u, y) \cdot v \right\| = 0. \tag{5.3}$$

Moreover, by mean value theorem,

$$\begin{aligned}
&\frac{1}{\lambda} [F(u + \lambda v, y) - F(u, y)] - D_u F(u, y) \cdot v \\
&= \int_0^1 [D_u F(u + \lambda \theta v) - D_u F(u, y)] \cdot v d\theta
\end{aligned}$$

so that, due to the boundedness of $D_u F(u, y)$, we get

$$\left\| \frac{1}{\lambda} [F(u + \lambda v, y) - F(u, y)] - D_u F(u, y) \cdot v \right\| \leq C \|v\|.$$

and then, by using the dominated convergence theorem, taking (5.3) into account, we can conclude

$$\begin{aligned}
&\lim_{\lambda \rightarrow 0} \left(\int_H \frac{1}{\lambda} [F(u + \lambda v, y) - F(u, y)] \mu(dy), w \right)_H \\
&= \int_H \left(D_u F(u, y) \cdot v, w \right)_H \mu(dy),
\end{aligned}$$

which implies that

$$\left(D\bar{F}(u) \cdot v, w \right)_H = \int_H \left(D_u F(u, y) \cdot v, w \right)_H \mu(dy).$$

□

Remark 5.1. As a consequence of Lemma 5.2, it is easily to check that

$$\left(D_x \bar{\mathbf{F}}(x) \cdot h, k \right)_{\mathcal{H}} = \int_H \left(D_x \mathbf{F}(x, y) \cdot h, k \right)_{\mathcal{H}} \mu(dy), \quad h, k \in \mathcal{H}, \tag{5.4}$$

and this yields

$$\begin{aligned}
\|D_x \bar{\mathbf{F}}(x) \cdot h\| &\leq \int_H \|D_x \mathbf{F}(x, y) \cdot h\| \mu(dy) \\
&= \int_H \|A^{-\frac{1}{2}} [(D_u F)(\Pi_1 \circ x, y) \cdot (\Pi_1 \circ h)]\| \mu(dy) \\
&\leq C \int_H \|(D_u F)(\Pi_1 \circ x, y) \cdot (\Pi_1 \circ h)\| \mu(dy).
\end{aligned}$$

Moreover, by invoking conditions (2.5), we get

$$\|D_x \bar{\mathbf{F}}(x) \cdot h\| \leq C \|h\|, \quad h \in \mathcal{H}. \quad (5.5)$$

As far as the higher order derivative are concerned, by proceeding as in the proof of above lemma, we can show that

$$\left(D_{uu}^2 \bar{F}(u) \cdot (v, w), \nu \right)_H = \int_H \left(D_{uu} F(u, y) \cdot (v, w), \nu \right)_H \mu(dy), \quad v, w, \nu \in H.$$

As a consequence, we obtain

$$\left(D_{xx}^2 \bar{\mathbf{F}}(x) \cdot (h, k), l \right)_{\mathcal{H}} = \int_H \left(D_{xx}^2 \mathbf{F}(x, y) \cdot (h, k), l \right)_{\mathcal{H}} \mu(dy), \quad h, k, l \in \mathcal{H} \quad (5.6)$$

and

$$\|D_{xx}^2 \bar{\mathbf{F}}(x) \cdot (h, k)\| \leq C \|h\| \cdot \|k\|, \quad h, k \in \mathcal{H}. \quad (5.7)$$

Lemma 5.3. *For any $T > 0$, there exists $C_T > 0$ such that for any $x \in \mathcal{H}$ and $t \in [0, T]$, we have*

$$\|D_x \bar{u}(t, x)\| \leq C_T.$$

Proof. Note that for any $h \in \mathcal{H}$,

$$\begin{aligned} D_x \bar{u}(t, x) \cdot h &= \mathbb{E} \left[D\Phi(\bar{X}_t(x)) \cdot \eta_t^{h,x} \right] \\ &= \mathbb{E} \left(\Phi'(\bar{X}_t(x)), \eta_t^{h,x} \right)_{\mathcal{H}}, \end{aligned}$$

where $\eta_t^{h,x}$ is the mild solution of

$$\begin{cases} d\eta_t^{h,x} = \left(\mathcal{A}\eta_t^{h,x} + D\bar{\mathbf{F}}(\bar{X}_t(x)) \cdot \eta_t^{h,x} \right) dt \\ \eta^{h,x}(0) = h. \end{cases}$$

This means that $\eta_t^{h,x}$ is the solution of the integral equation

$$\eta_t^{h,x} = \mathcal{S}_t h + \int_0^t \mathcal{S}_{t-s} [D\bar{\mathbf{F}}(\bar{X}_s(x)) \cdot \eta_s^{h,x}] ds,$$

and then thanks to (5.5), we get

$$\|\eta_t^{h,x}\| \leq \|h\| + C \int_0^t \|\eta_s^{h,x}\| ds.$$

Then by Gronwall lemma it follows that

$$\|\eta_t^{h,x}\| \leq C_T \|h\|, \quad t \in [0, T], \quad (5.8)$$

which means

$$|D_x \bar{u}(t, x) \cdot h| \leq C_T \sup_{z \in \mathcal{H}} |\Phi(z)| \cdot \|h\|,$$

so that

$$\|D_x \bar{u}(t, x)\| \leq C_T.$$

□

Lemma 5.4. *For any $T > 0$, there exists $C_T > 0$ such that for any $x, h, k \in \mathcal{H}$ and $t \in [0, T]$, we have*

$$|D_{xx}^2 \bar{u}(t, x) \cdot (h, k)| \leq C_{T, \phi} \|h\| \cdot \|k\|.$$

Proof. For any $h, k \in \mathcal{H}$, we have

$$\begin{aligned} D_{xx}^2 \bar{u}(t, x) \cdot (h, k) &= \mathbb{E}[\Phi''(\bar{X}_t(x)) \cdot (\eta_t^{h,x}, \eta_t^{k,x}) \\ &\quad + \Phi'(\bar{X}_t(x)) \cdot \zeta_t^{h,k,x}], \end{aligned} \quad (5.9)$$

where $\zeta^{h,k,x}$ is the mild solution of equation

$$\begin{cases} d\zeta_t^{h,k,x} = [\mathcal{A}\zeta_t^{h,k,x} + \bar{\mathbf{F}}''(\bar{X}_t(x)) \cdot (\eta_t^{h,x}, \eta_t^{k,x}) + \bar{\mathbf{F}}'(\bar{X}_t(x)) \cdot \zeta_t^{h,k,x}] dt \\ \zeta_0^{h,k,x} = 0. \end{cases}$$

This means that $\zeta_t^{h,k,x}$ is the solution of the integral equation

$$\zeta_t^{h,k,x} = \int_0^t \mathcal{S}_t[\bar{\mathbf{F}}''(\bar{X}_s(x)) \cdot (\eta_s^{h,x}, \eta_s^{k,x}) + \bar{\mathbf{F}}'(\bar{X}_s(x)) \cdot \zeta_s^{h,k,x}] ds.$$

Thus, by (5.5) and (5.7) we have

$$\begin{aligned} \|\zeta_t^{h,k,x}\| &\leq C \int_0^t (\|\eta_s^{h,x}\| \cdot \|\eta_s^{k,x}\| + \|\zeta_s^{h,k,x}\|) ds \\ &\leq C \|h\| \cdot \|k\| + C \int_0^t \|\zeta_s^{h,k,x}\| ds. \end{aligned}$$

By applying the Gronwall lemma we have

$$\|\zeta_t^{h,k,x}\| \leq C_T \|h\| \cdot \|k\|, \quad t > 0.$$

Returning to (5.9), we can get

$$|D_{xx} \bar{u}(t, x) \cdot (h, k)| \leq C \|h\| \cdot \|k\|. \quad (5.10)$$

□

By proceeding again as in the proof of above lemma, we have the following result.

Lemma 5.5. *For any $T > 0$, there exists $C_T > 0$ such that for any $x, h, k, l \in \mathcal{H}$ and $t \in [0, T]$, we have*

$$D_{xxx}^3 \bar{u}(t, x) \cdot (h, k, l) \leq C_{T, \phi} \|h\| \cdot \|k\| \cdot \|l\|.$$

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